

# A lenient BKR operation for events occurring for disjoint reasons

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## Abstract

Given events  $A$  and  $B$  on the product space  $S = \prod_{i=1}^n S_i$ , the set  $A \square B$  consists of all vectors  $\mathbf{x} = (x_1, \dots, x_n) \in S$  for which there exist disjoint coordinate subsets  $K$  and  $L$  of  $\{1, \dots, n\}$  such that given the coordinates  $x_i, i \in K$  one has that  $\mathbf{x} \in A$  regardless of the values of the remaining coordinates, and likewise that  $\mathbf{x} \in B$  given the coordinates  $x_j, j \in L$ . For a product of discrete spaces endowed with product measure, the BKR inequality

$$P(A \square B) \leq P(A)P(B) \quad (1)$$

was conjectured by van den Berg and Kesten [3] and proved by Reimer [10].

In [4] inequality (1) was extended to general product probability spaces, replacing  $A \square B$  by the set  $A \square\!\!\!\square B$  consisting of those outcomes  $\mathbf{x}$  for which one can assure only with probability one that  $\mathbf{x} \in A$  and  $\mathbf{x} \in B$  based only on the revealed coordinates in  $K$  and  $L$  as above. A strengthening of the original BKR inequality (1) results, due to the fact that  $A \square B \subset A \square\!\!\!\square B$ . In particular, it may be the case that  $A \square B$  is empty, while  $A \square\!\!\!\square B$  is not.

We propose the further extension  $A \square\!\!\!\square\!\!\!\square B$  depending on probability thresholds  $s$  and  $t$ , where  $A \square\!\!\!\square B$  is the special case where both  $s$  and  $t$  take the value zero. The outcomes  $\mathbf{x}$  in  $A \square\!\!\!\square\!\!\!\square B$  are those for which disjoint sets of coordinates  $K$  and  $L$  exist such that given  $\mathbf{x}$  on the revealed set of coordinates the probability that events  $A$  and  $B$  do not occur is bounded above by  $s$  and  $t$ , respectively. We provide simple examples that illustrate the utility of these extensions.

## 1 Introduction

The ‘box’  $A \square B$  of events  $A$  and  $B$ , consisting of the set of outcomes where  $A$  and  $B$  ‘happen for different reasons’ was first considered in [11] on the space  $S = \{0, 1\}^n$  with the uniform probability measure. Formally, for  $K \subset \{1, \dots, n\}$ , defining the  $K$  cylinder of  $\mathbf{x} \in S$  as

$$[\mathbf{x}]_K = \{\mathbf{y} \in S : y_i = x_i \ \forall i \in K\},$$

the box operation is given by

$$A \square B = \{\mathbf{x} \in S : \text{there exist disjoint } K, L \subset \{1, \dots, n\} \text{ such that } [\mathbf{x}]_K \subset A, [\mathbf{x}]_L \subset B\}. \quad (2)$$

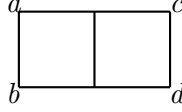
The set  $A \square B$  represents the event that  $A$  and  $B$  occur for disjoint reasons, as  $A$  can be verified to occur based only on the revealed values of the coordinates  $K$  of the outcome, regardless of the values of the remaining coordinates, and similarly so for  $B$  based on a set  $L$  of coordinates disjoint to  $K$ . Clearly  $A \square B \subset A \cap B$ . Note that if  $K \subset M \subset \{1, \dots, n\}$  then  $[\mathbf{x}]_M \subset [\mathbf{x}]_K$ . This implies readily that in (2) we can assume without loss of generality that  $L = \overline{K}$ , the complement of  $K$ . Note also that the choice of  $K$  and  $L$  may depend on  $\mathbf{x}$ , and we may write  $K(\mathbf{x})$  and  $L(\mathbf{x})$  for specific choices that satisfy the condition in (2).

The famous BKR inequality, conjectured in [11] and proved in [10] for the uniform probability measure on  $\{0,1\}^n$ , states that

$$P(A \square B) \leq P(A)P(B). \quad (3)$$

The validity of (3) for the uniform measure on  $\{0,1\}^n$  was shown to imply the inequality on a finite product of discrete probability spaces in [12].

**Example 1.1** (A simple example: bond percolation.). *Consider the following graph:*



*Edges are open or closed independently with probability 1/2. Since there are 7 edges we have  $S = \{0,1\}^7$ . Let  $A = \{\text{there exists a path of open edges from } a \text{ to } c\}$ . A direct count of the events that corresponds to  $A$  shows that under the uniform distribution,  $P(A) = 44/2^7$ . For  $B = \{\text{there exists a path of open edges from } b \text{ to } d\}$  it is easy to see that  $P(A \square B) = 2^3/2^7 = 1/16$  as  $A \square B$  is the set of all paths for which all bonds on the top and bottom rows of the graph are open.*

*As  $P(A)P(B) = (44/2^7)^2$ , in this case ‘the excess multiple’ in the bound (3) is given by*

$$\frac{P(A)P(B)}{P(A \square B)} = \frac{(44/2^7)^2}{1/16} \approx 1.9. \quad (4)$$

We now turn to general spaces. The function version (5) below of inequality (3) was proved in [4], see also [5], implying (3) for finite products of arbitrary probability spaces. Henceforth we take  $S = \prod_{i=1}^n S_i$ , a finite product of general spaces  $S_i, i = 1, \dots, n$  endowed with a product measure  $P = \prod_{i=1}^n P_i$  on the product sigma algebra of  $S$  generated by given sigma algebras on  $S_i$ . The case  $S_i = \mathbb{R}$ , for a finite and countable product was considered in [1], with a certain emphasis on issues of measurability. For an interesting practical application see [2].

## 2 The $\square$ Box operation

With the notation given further meaning later, we introduce the box type operation  $A \square B$  on events  $A$  and  $B$  that takes account of probability zero events in a natural probabilistic way. We first briefly describe some results from [4], here restricting attention to Borel measurable functions; for extensions to completions see the Appendix of [4]. Let  $f$  and  $g$  be two given non-negative Borel measurable functions on  $S$ . For  $K$  and  $L$  subsets of  $\{1, \dots, n\}$ , let

$$\underline{f}_K(\mathbf{x}) = \text{ess inf}_{\mathbf{y} \in [\mathbf{x}]_K} f(\mathbf{y}), \quad \text{and} \quad \underline{g}_L(\mathbf{x}) = \text{ess inf}_{\mathbf{y} \in [\mathbf{x}]_L} g(\mathbf{y}), \quad (5)$$

where the essential infimums defining  $\underline{f}_K(\mathbf{x})$  and  $\underline{g}_L(\mathbf{x})$  are taken with respect to the product probability measure on the coordinates in the complements  $\bar{K}$  and  $\bar{L}$  respectively.

**Theorem 2.1.** *Let  $\mathbf{X} = (X_1, \dots, X_n) \in S$  be a random vector with a product distribution  $P$ . Then*

$$E \left\{ \max_{K \cap L = \emptyset} \underline{f}_K(\mathbf{X}) \underline{g}_L(\mathbf{X}) \right\} \leq E \{f(\mathbf{X})\} E \{g(\mathbf{X})\}. \quad (6)$$

We remark that Theorem 2.1 applies to functions of a vector  $\mathbf{X}$  having independent coordinates taking values in *any* measure spaces.

Specializing (6) to the case where  $f$  and  $g$  are the indicator functions  $\mathbf{1}_A(\mathbf{x})$  and  $\mathbf{1}_B(\mathbf{x})$  of  $A$  and  $B$  respectively, we define  $A \sqcap B$  by its indicator

$$\mathbf{1}_{A \sqcap B} = \max_{K \cap L = \emptyset} \mathbf{1}_{A_K}(\mathbf{x}) \mathbf{1}_{B_L}(\mathbf{x}), \quad (7)$$

and now invoking Theorem 2.1 we have

$$P(A \sqcap B) \leq P(A)P(B). \quad (8)$$

Adopting definition (2) for  $A \sqcap B$  in general spaces, we note in contrast to (2), that the functions defined in (5), and hence the set  $A \sqcap B$ , depend on the underlying measure. As

$$P(A \sqcap B) \leq P(A \sqcap B), \quad (9)$$

inequality (8) is stronger than (3). To see (9), note that replacing essential infimum by infimum in (5), inequality (9) becomes equality. Hence (9) holds as stated because the essential infimum is at least as large as the infimum. In words, elements of  $A \sqcap B$  demand disjoint ‘reasons’ for  $A$  and  $B$  that hold for all outcomes in the probability space, while  $A \sqcap B$  requires only that the ‘reasons’ be almost sure.

To develop some intuition on the  $\sqcap$  box type operation, consider two individuals, one of whom desires that event  $A$  happens, while the other desires  $B$ ; we label the individuals according to their desired event with little danger of confusion. If  $\mathbf{x} \in A \sqcap B$  then disjoint sets of coordinates of  $\mathbf{x}$  may be revealed to  $A$  and  $B$  so that each will know with certainty that their event has occurred. Definition (7) extends the box operation to accommodate situations where the two individuals may be revealed disjoint sets of coordinates that indicate that their event has occurred almost surely, that is, with probability one, but may not certify it absolutely.

It is natural to ask what might be the differences, and possible advantages, of defining the box operation  $\sqcap$  as in (7), and more generally for functions as in (5) as applied in Theorem 2.1, where the underlying measure is involved, in contrast to the more ‘straightforward’ extension that preserves (2) as a pure set operation. For one, the approach taken in Theorem 2.1 avoids certain subtle measurability difficulties that arise in the ‘straightforward’ approach. In particular, Example 2 in [1] presents a situation where the classical box of two Lebesgue measurable sets fails to be Lebesgue measurable. In contrast, as shown in [4], the function operation as defined in (5) preserves measurability.

Technical matters aside, the approach of Theorem 2.1 also has advantages from a probabilistic perspective, as illustrated by the following simple example, somewhat related to Example 2 in [1]. Let

$$S = [0, 1]^2 \quad \text{and} \quad A = B = \{(x_1, x_2) \in [0, 1]^2 : x_1 \neq x_2\}. \quad (10)$$

It is easy to see that  $A \sqcap B$  as defined in (2) is empty. For instance, for any  $\mathbf{x} \in [0, 1]^2$ , the cylinder  $[\mathbf{x}]_{\{1\}}$  is not contained in  $A$  (only) because it contains the point  $(x_1, x_1)$ , which is not in  $A$ . However, for any continuous measure on  $S$ , this single point is a set of measure zero, with the result that  $A \sqcap B = [0, 1]^2$ .

Here is an intuitive way to think about this example: suppose Alice and Bob each need to choose a single instant of time in the interval  $[0, 1]$  required for completion of a certain task. Assume there is only a single resource, and both will fail if and only if they require the resource at the same precise instant. To succeed for disjoint reasons as in (2) means that each one can be sure of their own success independently of the other’s choice of time. That here  $A \sqcap B = \emptyset$  is a reflection of the fact that this is impossible. However, if they select times independently by any continuous distribution in the unit interval, then each one of them will succeed with probability one for any choice of the

other. And indeed, for this case  $A \overline{\square} B = [0, 1]^2$ . This example, though simple, illustrates how sets of measure zero, which typically are a technical nuisance that can effectively be ignored, can greatly affect box type operations, here in particular causing the maximum possible discrepancy in the inequality

$$0 = P(\emptyset) = P(A \square B) < P(A \overline{\square} B) = 1.$$

**Example 2.1** (Continuous percolation). *Consider the random geometric graph on a collection of  $n$  independent points  $\mathbf{X} = (X_1, \dots, X_n)$ , each sampled from a continuous distribution on  $[0, 1]^2$ , say. The vertex set of the graph is  $\mathbf{X}$ , and an edge is drawn between distinct points  $X_i$  and  $X_j$  if  $\|X_i - X_j\| \leq 2r$ , where  $r > 0$  and  $\|\cdot\|$  is any norm in  $\mathbb{R}^2$ . An equivalent way to view the graph is to connect two distinct points when the two circles of radius  $r$  having these points as their centers intersect. We note that the graph just defined is not planar in general. For models of this kind see [9] and [8].*

We now consider events that can be seen as continuous versions of events such as those studied in [3]. Let  $A$  be the event that there exists a ‘path along edges from left to right’, that is, a path starting from a point that is within distance less than  $r$  from the  $y$ -axis, and ending at a point within distance less than  $r$  from the line  $x = 1$ . Similarly, let  $B$  denote the event that there exists a path from bottom to top. Since the graph is not planar, these events could occur on two distinct paths, that is, paths that do not share a vertex. In this case  $A$  and  $B$  occurring for disjoint reasons coincides with the event  $A \square B$ .

We also consider this model with annihilation. If for some  $i \neq j$  the points  $X_i$  and  $X_j$  coincide, that is, are equal, then they annihilate each other. Processes with annihilation are ubiquitous, appearing in connections to random walks, Brownian motion, branching processes and more, based on motivations from physics and chemistry. We consider the case where  $X_i, i = 1, \dots, n$  are independent, each with a continuous distribution, in which case the probability of annihilation is zero. That  $A \square B = \emptyset$  in this example is covered by the following more general result.

**Proposition 2.1.** *Adding annihilation to the percolation model described in Example 2.1,  $A \square B = \emptyset$  where  $A$  and  $B$  are any events that require that at least one point is not annihilated.*

*Proof.* For any such event  $A$ , any  $\mathbf{x}$  and any index set  $K$  such that  $|\overline{K}| \geq |K|$ , it is impossible that  $[\mathbf{x}]_K \subset A$  as the  $|\overline{K}|$  unspecified coordinates in  $[\mathbf{x}]_K$ , those indexed by the larger set  $\overline{K}$ , can annihilate all the points indexed by  $K$ . Hence  $A \square B$  must be empty, as given complementary sets  $K$  and  $L$  at least one set has a size dominated by the size of its complement.  $\square$

We remark that Proposition 2.1 has a flavor similar to the example based on (10), in that the outcome of the classical box operation, due to its definition, may be determined by events that will never occur.

The following result is obvious since annihilation occurs with probability zero.

**Proposition 2.2.** *The event  $A \square B$  for the model without annihilation is contained in the event  $A \overline{\square} B$  for the model with annihilation.*

*Proof.* Suppose  $\mathbf{x} \in A \square B$  for the model without annihilation. By definition there exist disjoint  $K, L \subset \{1, \dots, n\}$  such that  $[\mathbf{x}]_K \subset A, [\mathbf{x}]_L \subset B$ . For the model with annihilation, since annihilation occurs with probability zero, we have  $P(A \cap [\mathbf{x}]_K) = 1$  and  $P(B \cap [\mathbf{x}]_L) = 1$ , and it follows from (7) that  $\mathbf{x} \in A \overline{\square} B$ .  $\square$

In many natural examples,  $A \square B \neq \emptyset$  for the model without annihilation. However, by Proposition 2.1  $A \square B = \emptyset$  for non trivial sets under the model with annihilation. Proposition 2.2 implies in particular that if  $A \square B \neq \emptyset$  for the model without annihilation then  $A \overline{\square} B \neq \emptyset$  with annihilation.

### 3 Cylindrification

Before proceeding to our extension of the box operation allowing non-zero probability tolerances, we pause to make a simple observation about potential improvements to the original box inequality, and our subsequent generalizations. Here we show that ‘cylindrical cores’ of the sets that comprise the box operation may be separately extracted, yielding no loss to the left-hand side of (3), but a possibly smaller quantity on the right. In particular, given a set  $E$ , let the cylindrical core of  $E$  be given by

$$E^0 = \bigcup_{|K| \leq n-1, [\mathbf{x}]_K \subset E} [\mathbf{x}]_K. \quad (11)$$

One can immediately verify that  $E^0 \subset E$ . In Proposition 3.1 we show that  $E^0$  is the smallest set that preserves the result of the box operation applied to  $E$  and any proper subset of  $S$ . Applying this result to any proper subsets  $A$  and  $B$  of  $S$ , we obtain, for possible improvements in (3),

$$A^0 \subset A, \quad B^0 \subset B \quad \text{and} \quad A \square B = A^0 \square B^0.$$

By symmetry, we may focus only on  $A$ . We exclude the trivial case of  $S$  in (12) since  $A \square S = A$  for all  $A$ , for which  $A \square S = A^0 \square S$  implies  $A = A^0$ , precluding the possibility of having the equality (12) hold for all  $B$  for any proper subset of  $A$ .

**Proposition 3.1.** *Let  $A \subset S$ . Then the set  $A^0$  defined in (11) is the smallest set that satisfies*

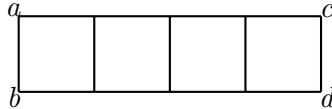
$$A \square B = A^0 \square B \quad \forall B \neq S \quad (12)$$

*in the sense that if (12) is satisfied for a set  $A^1$  in place of  $A^0$ , then  $A^1 \supset A^0$ .*

*Proof:* First it is clear that  $A^0 \subset A$ , as  $A^0$  is a union of subsets of  $A$ . Hence, by the evident monotonicity of the box operation, in order to prove (12) we are only required to show that  $A \square B \subset A^0 \square B$  for all  $B \neq S$ . If  $\mathbf{x} \in A \square B$  then there exist disjoint  $K$  and  $L$  such that  $[\mathbf{x}]_K \subset A$  and  $[\mathbf{x}]_L \subset B$ . Note that  $B \neq S$  implies  $|L| \geq 1$  and hence  $|K| \leq n - 1$ . Now by definition  $[\mathbf{x}]_K \subset A^0$ , and  $\mathbf{x} \in A^0 \square B$ .

It remains to prove the minimality of  $A^0$ . If  $\mathbf{x} \in A^0$  then for some  $K$  satisfying  $|K| \leq n - 1$  we have  $[\mathbf{x}]_K \subset A^0$ . Now, with  $L = \overline{K}$  let  $B = [\mathbf{x}]_L$ . As  $|L| = n - |K| \geq 1$ , we have that  $B \neq S$  and therefore  $\mathbf{x} \in A^0 \square B = A^1 \square B$ , so in particular  $\mathbf{x} \in A^1$ . The inclusion  $A^0 \subset A^1$  is hence demonstrated.  $\square$

**Example 3.1** (Bond Percolation). *Consider bond percolation on the graph below. There are 13 edges, each of which is open or closed independently, and  $|S| = 2^{13}$ . A path from left to right in this example means a path from either  $a$  or  $b$  to either  $c$  or  $d$  along open edges.*



*Let  $A = \{\text{there exists a path from left to right}\} \cup \{\text{all edges are closed}\} \cup \{\text{all edges but exactly two are closed}\}$ . In this case  $A^0 = \{\text{there exists a path from left to right}\}$  which is strictly a subset of  $A$ . More specifically,  $A \setminus A^0$  contains  $1 + \binom{13}{2}$  points. Assuming, for example, a uniform probability, we obtain  $P(A) - P(A^0) = (1 + \binom{13}{2})/2^{13}$ .*

## 4 The lenient $s, t$ box

In Section 1 we introduced the box operation  $\boxed{oo}$  that is not influenced by events of zero probability. Here we embed this operation as the special case  $(s, t) = (0, 0)$  of the  $\boxed{st}$  operation where ‘leniency’ of conditional probabilities of size  $s$  and  $t$  are granted to the events  $A$  and  $B$  respectively in (13). Formally, for  $A, B$  subsets of  $S$  and  $s, t \in [0, 1]$  let

$$A\boxed{st}B = \{\mathbf{x} \in S : \exists \text{ disjoint } K, L \subset \{1, \dots, n\} \text{ such that } P(\overline{A} \mid [\mathbf{x}]_K) \leq s, P(\overline{B} \mid [\mathbf{x}]_L) \leq t\}. \quad (13)$$

In words, the event  $A\boxed{st}B$  consists of all outcomes  $\mathbf{x}$  for which there are disjoint sets of coordinate indices  $K$  and  $L$  such that the conditional probabilities of  $A$  given the values of  $x_i$  for  $i \in K$ , and  $B$  given the values of  $x_j$  for  $j \in L$ , are at least  $1 - s$  and  $1 - t$ , respectively. In contrast to definition (2), here we cannot assume without loss of generality that the sets of indices  $K$  and  $L$  are complements; see Example 5.1 for an illustration.

The next proposition shows that definitions (13) and (7) are consistent, that is, that they coincide when  $(s, t) = (0, 0)$ , and in particular explains our earlier choice of notation for  $A\boxed{oo}B$ .

**Proposition 4.1.** *For  $(s, t) = (0, 0)$  we have  $A\boxed{st}B = A\boxed{oo}B$ .*

*Proof.* By (7),  $\mathbf{x} \in A\boxed{oo}B$  if and only if  $\text{ess inf}_{\mathbf{y} \in [\mathbf{x}]_K} \mathbf{1}_A(\mathbf{y}) = \text{ess inf}_{\mathbf{y} \in [\mathbf{x}]_L} \mathbf{1}_B(\mathbf{y}) = 1$  for some disjoint  $K$  and  $L$ , if and only if  $P(\overline{A} \mid [\mathbf{x}]_K) = P(\overline{B} \mid [\mathbf{x}]_L) = 0$  for some disjoint  $K$  and  $L$ , if and only if  $\mathbf{x} \in A\boxed{st}B$ .  $\square$

Theorem 4.1 gives a version of the BKR inequality for the lenient box operation, for which the following definition is needed. For a subset  $E \subset S$  and  $r \in [0, 1]$  define the ‘ $r$  inflated  $E$  set’ by

$$E_r = \{\mathbf{x} \in S : \exists J \subset \{1, \dots, n\} \text{ such that } P(\overline{E} \mid [\mathbf{x}]_J) \leq r\}. \quad (14)$$

**Theorem 4.1.** *If  $A, B$  are events on  $S$ , then for all  $s, t \in [0, 1]$*

$$P(A\boxed{st}B) \leq P(A_s)P(B_t).$$

*Proof:* It suffices to show  $A\boxed{st}B \subset A_s \square B_t$ , as then the result follows by applying inequality (3). If  $\mathbf{x} \in A\boxed{st}B$  then there exist disjoint  $K$  and  $L$  such that

$$P(\overline{A} \mid [\mathbf{x}]_K) \leq s \quad \text{and} \quad P(\overline{B} \mid [\mathbf{x}]_L) \leq t.$$

In particular,  $[\mathbf{x}]_K \subset A_s$  and  $[\mathbf{x}]_L \subset B_t$ , and as  $K$  and  $L$  are disjoint we have  $\mathbf{x} \in A_s \square B_t$  as desired.  $\square$

We note that inequality (8), and hence also (3), follow from the special case  $s = t = 0$  of Theorem 4.1.

## 5 Lenient box examples

**Example 5.1** (An odd coin tossing problem). *A fair coin is to be tossed independently  $n = 2m + 1$  times. Let  $A$  be the event that the first  $m + 1$  tosses are all heads, and  $B$  the event that the last  $m + 1$  tosses are all tails. With  $s = t = 1/2$ , we claim that  $A\boxed{st}B$  consists of the two sequences in which the first  $m$  tosses are heads, the last  $m$  tosses are tails and the middle toss could be either.*

*To verify the claim, note that when  $\mathbf{x} \in A\boxed{st}B$  there must exist  $K$  such that  $P(\overline{A} \mid [\mathbf{x}]_K) \leq 1/2$ , and hence  $x$  must specify an outcome where  $A$  fails to occur with probability at most  $1/2$ . In other words,  $x$  restricted to  $K$  must specify that either  $A$  has occurred, or an outcome consistent with  $A$  in which a single toss needed to achieve  $A$  has not been specified. Similar reasoning applies to  $\mathbf{x}$  as*

regards  $B$ , and the two stated outcomes are the only ones consistent with both  $A$  and  $B$  where each event is lacking a single, unspecified toss. Hence

$$P(A[\overline{st}]B) = 2 \times (1/2)^n = (1/2)^{2m}.$$

The  $1/2$  inflated  $A$  set  $A_{1/2}$  as defined in (14) consists of  $A$ , and the sequences that have exactly one 0 in the first  $m+1$  positions. Hence

$$P(A_s) = (1/2)^{m+1} + (m+1)(1/2)^{m+1} = (m+2)(1/2)^{m+1}.$$

Arguing similarly for  $B$ , the inequality

$$(1/2)^{2m} = P(A[\overline{st}]B) \leq P(A_{1/2})P(B_{1/2}) = ((m+2)(1/2)^{m+1})^2 = (m+2)^2(1/2)^{2m+2}$$

verifies the conclusion of Theorem (4.1). These calculations show that in this case, the excess multiple in the bound, such as the one computed in (4) for the standard box, is given by

$$\frac{P(A_{1/2})P(B_{1/2})}{P(A[\overline{st}]B)} = \left(\frac{m+2}{2}\right)^2.$$

In this case the lenient box is not empty, yet not only is the box empty, so also is the intersection, that is  $P(A \cap B) = \emptyset$ .

**Example 5.2** (Continuous percolation with positive-probability annihilation). In Proposition 2.1 following Example 2.1, a continuous percolation model was considered that included annihilation occurring with probability zero. There, distinct points  $X_i$  and  $X_j$  annihilate each other when they coincide, that is, when they are within distance  $q = 0$ . We now generalize this model to one where distinct points annihilate when they fall within distance  $0 \leq q < r$  of each other.

Assume that  $X_1, \dots, X_n$  are i.i.d. with density bounded by  $c$  on  $[0, 1]^2$ . When  $q > 0$  the probability that any given point will be annihilated is positive, and hence it is easy to see that for  $A$  and  $B$  the events given in Example 2.1 of Section 2, we have  $A[\overline{oo}]B = \emptyset$ . The next proposition shows that when the thresholds  $s$  and  $t$  are sufficiently large then  $A[\overline{st}]B$  is non-empty, and identifies some of its elements. To simplify matters, we now restrict to metrics in  $\mathbb{R}^2$  that are induced by norms. In particular, under this assumption the area of balls of radius  $r > 0$  centered at  $x \in \mathbb{R}^2$  depends only on  $r$ .

**Proposition 5.1.** Let  $\mathbf{x} = (x_1, \dots, x_n) \in \{[0, 1]^2\}^n$  be such that for some  $K \subset \{1, \dots, n\}$  the points  $\{x_i : i \in K\}$  form a path on edges from left to right corresponding to the event  $A$ , in spite of possible annihilation among this same set of points, and similarly, for a disjoint set  $L$  and a path corresponding to  $B$ . Then  $\mathbf{x} \in A[\overline{st}]B$  provided  $\{s, t\} \subset [\frac{c}{4}n^2\pi_q, 1]$ , where  $\pi_q$  denotes the area of a circle of radius  $q$ .

*Proof.* Consider  $A$ , reasoning for  $B$  being similar, and take  $\mathbf{x}$  and  $K$  such that  $\{x_i, i \in K\}$  forms a path, possibly in spite of some annihilation among these same points. The number of points that make up such a path is obviously at most  $|K|$ , and hence there is at most area  $|K|\pi_q$  into which a point of  $\overline{K}$  can land to cause annihilation, which happens with probability at most  $|K|c\pi_q$ . Hence, letting  $E_j$  be the event that the point indexed by  $j \in \overline{K}$  annihilates some point on the path, the conditional probability of the event  $F$  that some point on the path is annihilated, given  $\{x_i, i \in K\}$  satisfies

$$P(F \mid [\mathbf{x}]_K) = P(\cup_{j \in \overline{K}} E_j \mid [\mathbf{x}]_K) \leq \sum_{j \in \overline{K}} P(E_j \mid [\mathbf{x}]_K) \leq (n - |K|)|K|c\pi_q \leq \frac{c}{4}n^2\pi_q \leq s. \quad (15)$$

Given the occurrence of the points indexed by  $K$ , ‘which already’ form a path, event  $A$  can only fail to occur if some point on this path is annihilated. Hence  $P(\bar{A} \mid [\mathbf{x}]_K) \leq P(F \mid [\mathbf{x}]_K) \leq s$ . Repeating the argument for  $B$ , the result follows in view of (13).  $\square$

An intuitive way to think about the situation considered in Proposition 5.1 is to consider the case where points indexed by  $K$  have been observed, and, perhaps despite annihilation among the observed points, form a path corresponding to  $A$ . We then ask about the conditional probability that the remaining, unobserved points will destroy this path by annihilation. It is easy to see that the probability that at least one unobserved point annihilates an observed one is at most  $1 - (1 - |K|c\pi_q)^{n-|K|} \leq (n - |K|)|K|c\pi_r$ , agreeing with the bound in (15).

Note that  $r$  of Example 2.1 can be chosen so small that  $\lceil n/2 \rceil$  points may not suffice to form a path as required by the events  $A$  and  $B$ . In this case disjoint  $K$  and  $L$ , both of which must demonstrate paths, cannot exist. Nevertheless, in such a case  $A \square_{st} B$  may still be non-empty, as the coordinates of  $\mathbf{x}$  revealed by  $K$  and  $L$  when considered only marginally may be sufficiently promising for the chances that the yet unrevealed coordinates will yield the desired outcome.

## 6 Earlier results on continuous spaces

We conclude with a brief mention of an earlier BKR result for the special case of continuous spaces given in [6]. The Poisson Boolean percolation model consists of an inhomogeneous Poisson point process in some bounded region  $U$  of  $\mathbb{R}^d$ , each with a ball of a random radius around it. For events  $A$  and  $B$  defined by this process,  $A \square B$  comprises those configurations of process points for which the occurrence of  $A$  and  $B$  can respectively be claimed upon revealing all process points lying in two disjoint spatial regions. As this definition includes the case (2) where two sets of points having disjoint indices are revealed,  $A \square B$  as defined in [6] may be larger than that defined by (2). In this setting, inequality (3) is proved in a way that relies heavily on properties of the Poisson process. Therefore, it appears that the this Poisson process case does not imply our results for a fixed number of points.

Conversely, even leaving the spatial structure in [6] aside, it seems that our results do not lead to results that concern a random number of points. We can condition on the number of Poisson points and obtain  $P(A \square B \mid N) \leq P(A \mid N)P(B \mid N)$  and take expectation with respect to  $N$ . However, the inequality in  $E\{P(A \mid N)P(B \mid N)\} \leq E\{P(A \mid N)\}E\{P(B \mid N)\} = P(A)P(B)$  holds only under very special conditions on  $A$  and  $B$ .

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## References

- [1] Arratia R, Garibaldi S, Hales AW (2015) The van den Berg-Kesten-Reimer operator and inequality for infinite spaces. arXiv:1508.05337 [math.PR]
- [2] Arratia R, Garibaldi S, Mower L, Stark PB (2015) Some people have all the luck. arXiv:1503.02902 [math.PR]
- [3] van den Berg J, Kesten H (1985) Inequalities with applications to percolation and reliability. J. Appl. Probab. 22(3):556-569



- [4] Goldstein L, Rinott Y (2007) Functional BKR inequalities, and their duals, with applications. Jour. Theor. Probab. 20(2):275–293
- [5] Goldstein L, Rinott Y (2015) Functional van den Berg–Kesten–Reimer Inequalities and their Duals, with Applications. arXiv:1508.07267 [math.PR]
- [6] Gupta JC, Rao BV (1999) van den Berg–Kesten inequality for the Poisson Boolean Model for continuum Percolation. Sankhya A 61(3):337–346
- [7] Last G, Penrose MD, Zuyev S (2016) On the capacity functional of the infinite cluster of a Boolean model. arXiv:1601.04945 [math.PR]
- [8] Meester R, Roy R (2008) Continuum percolation. Cambridge University Press
- [9] Penrose, M. (2003). Random geometric graphs (No. 5). Oxford University Press.
- [10] Reimer D (2000) D. Reimer (2000) Proof of the Van den Berg–Kesten conjecture. Combin Probab Comput 9(1):27–32.
- [11] van den Berg J, Kesten H (1985) Inequalities with applications to percolation and reliability, J. Appl Probab 22(3):556–569
- [12] van den Berg J, Fiebig U (1987) On a combinatorial conjecture concerning disjoint occurrences of events, Ann. Probability 15(1) 354–374.